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Letter to the Editor

A Remark on Best L^1-Approximation by Polynomials

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Let P_n denote the set of all real polynomials of degree n. The sequence

$$E_n[f] := \inf_{p \in P_n} \int_{-1}^{1} |f(x) - p(x)|\, dx, \qquad n = 0, 1, ...$$

has been studied in the literature for various real functions $f \in L^1[-1, 1]$ [1, 4–7]. The starting point was mostly the following theorem of Markoff: Let

$$U_n(t) := \frac{\sin[(n+1)\arccos t]}{\sqrt{1-t^2}}$$

denote the Chebyshev polynomials of the second kind and let $\text{intpol}_n[f]$ denote the interpolation polynomial of f with respect to the zeros of U_n as nodes. If $(f - \text{intpol}_n[f])\, U_n$ has no changes of sign on $[-1, 1]$, then

$$E_{n-1}[f] = \left| \int_{-1}^{1} f(t) \operatorname{sgn} U_n(t)\, dt \right|. \tag{1}$$

The purpose of this note is to point out that the evaluation of the integral (1) is easy if the expansion of f in terms of U_ν,

$$f \sim \sum_{\nu=0}^{\infty} b_\nu U_\nu, \qquad b_\nu = \frac{2}{\pi} \int_{-1}^{1} f(x)\, U_\nu(x) \sqrt{1-x^2}\, dx,$$

or in terms of T_ν ($:= \cos \nu \operatorname{arcos} x$),

$$f \sim \frac{a_0}{2} + \sum_{\nu=1}^{\infty} a_\nu T_\nu, \qquad a_\nu = \frac{2}{\pi} \int_{-1}^{1} f(x) \frac{T_\nu(x)}{\sqrt{1-x^2}},$$

is known.

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THEOREM. *If* $f \in L^1[-1, 1]$, *then*

$$\int_{-1}^{1} f(t) \operatorname{sgn} U_n(t)\, dt = 2 \sum_{\nu=0}^{\infty} (2\nu+1)^{-1} b_{(2\nu+1)(n+1)-1}. \tag{2}$$

Proof. It is well known (e.g., [11, p. 90]) that the Fourier series

$$\operatorname{sgn} \sin x = 4\pi^{-1} \sum_{\nu=0}^{\infty} (2\nu+1)^{-1} \sin(2\nu+1)\, x$$

has uniformly bounded partial sums. Therefore

$$\operatorname{sgn} U_n(t) = 4\pi^{-1} \sum_{\nu=0}^{\infty} (2\nu+1)^{-1} \sin(n+1)(2\nu+1) \arccos t$$

has the same property, and the application of Lebesgue's dominated convergence theorem leads to

$$\begin{aligned}\int_{-1}^{1} & f(t) \operatorname{sgn} U_n(t)\, dt \\ &= 4\pi^{-1} \sum_{\nu=0}^{\infty} (2\nu+1)^{-1} \int_{-1}^{1} f(t) \\ &\quad \times \sin(n+1)(2\nu+1) \arccos t\, dt \\ &= 2 \sum_{\nu=0}^{\infty} (2\nu+1)^{-1} b_{(2\nu+1)(n+1)-1}.\end{aligned}$$

The coefficients a_ν were studied by many authors (e.g., [2, 3, 8–10]). Because of $2b_\nu = a_\nu - a_{\nu+2}$ we can apply (2) to a great variety of functions. The aforementioned results on $E_n[f]$ (via (1)) may be derived in a simple and uniform manner by using (2). Many further examples of the application of (1) and (2) are equally simple; the following special cases may be mentioned:

$$f(x) = (1-x^2)^{-1/2}, \qquad E_{2n-1}[f] = \pi(2n+1)^{-1},$$

$$f(x) = \arcsin x, \qquad E_{2n-2}[f] = \frac{\pi}{2n}\left(1 + \tan^2 \frac{\pi}{4n}\right) - 2 \tan \frac{\pi}{4n},$$

$$f(x) = (x^2+a^2)^{-1}, \qquad E_{2n-1}[f] = 4|a|^{-1} \arctan(\sqrt{a^2-1} - |a|)^n,$$

$$f(x) = |x|^s, \qquad s > -1 \text{ not an even integer},$$

$$E_{2n-1}[f] = \frac{8\Gamma(s+1)|\sin 2^{-1}\pi s|}{\pi(2n+1)^{s+1}} \left(\sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(2\nu+1)^{s+2}}\right)(1 + O(n^{-2})).$$

The application of Markoff's theorem is allowed; this can be shown using symmetry and Rolle's theorem. See the paper of Fiedler and Jurkat [4].

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